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A NOTE ON MINIMAL TRIANGULATIONS OF AN n-CUBE

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This paper is concerned with finding a lower bound for $\varphi(n)$, the minimum number of simplices required to triangulate an n-cube. We prove that $\varphi(n)\geq L_n$, where L_n is the minimum value of the sum of $n-1$ unknowns subject to $n-1$ inequality constraints. In particular, it is shown that $\varphi(5)\geq 60$.

1. Introduction

The purpose of this note is to further explore $\varphi(n)$, the minimum number of simplices required to triangulate an n-cube. This problem has been considered by Mara [3,4], Cottle [1], and Sallee [5]. We will prove that

Theorem. *If S is a triangulation of the n-cube whose vertices agree with those of the cube, then $|S|\geq L_n$, where L_n is the minimum of $x_2+x_3+\cdots+x_n$ subject to*

$$\frac{n!}{(n-k)!}x_2\frac{2!}{(n-k)!}h_2+\sum_{i=3}^{n-k}\frac{(n-i)!}{(n-i-k)!}x_i\frac{i!}{(n-k)!}h_i\geq\frac{2^k n!}{(n-k)!}$$

for $0\leq k\leq n-3$, $x_2\geq 2^{n-1}$ and h_n is defined by

$$h_n=\frac{1}{n!}\left\lfloor\frac{(n+1)^{(n+1)/2}}{2^n}\right\rfloor.$$

It follows immediately from the theorem that if we consider only triangulations whose vertices coincide with those of the cube, then $\varphi(n)\geq L_n$. As will be seen, $L_4=16$ and $L_5=60$. Thus we have a third proof [1,5] that $\varphi(4)=16$.

2. The inequalities

We use, without specific reference, a number of basic properties and definitions of (convex) polytopes. They can all be found in the book by Grünbaum [2].

In this paper, $I^n=[0,1]^n$. If $s\subset I^n$ is an n-simplex, a facet of s will be called

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exterior if it is a subset of a facet of I^n. I^n has $2n$ facets each of which is congruent to I^{n-1}. A finite set S of n-simplices is called a *triangulation* of I^n if $\bigcup S = I^n$ and the intersection of any two members of S is a face of each of them. The set of vertices of S is simply the set of vertices of elements of S.

Lemma 1. *If a simplex $s \subset I^n$ has k exterior facets, then each exterior facet has at least $k-1$ exterior facets in the $(n-1)$-cube in which it lies.*

Proof. Define $F_{ij}, i=1,2,\dots,n,\ j=0,1$ by

$$F_{ij} = \{x \in I^n \colon x_i = j\}.$$

It is clear that F_{ij} is a facet of I^n. Let s be an n-simplex contained in I^n, $\{v_i\} = \mathrm{vert}(s)$ be the set of vertices of s, and v_{ij} be the jth component of v_i. If s has a facet contained in F_{rm}, then $v_{ir} = m$ for n of the v_i's. It now follows that if s has k exterior facets, then the matrix (v_{ij}) has k columns whose entries contain n 0's or n 1's. If column r is such a column and (v'_{ij}) is the matrix obtained from (v_{ij}) by deleting the row whose rth entry differs from the other n, we see that (v'_{ij}) must have at least $k-1$ columns, excluding the rth column, whose entries contain either $n-1$ 0's or $n-1$ 1's. This proves the lemma.

Lemma 2. *For $n \geq 2$, any n-simplex $s \subset I^n$ has at most n exterior facets.*

Proof. This is obvious for if s had $n+1$ exterior facets, then some two of them would lie in opposite facets of I^n.

In this paper, if a is an m-dimensional object in n space, $V(a)$ is used to denote the m-dimensional volume of a.

Lemma 3. *For $n \geq k \geq 3$, if s is an n-simplex contained in I^n satisfying the inequality*

$$V(s) \geq \frac{(k-1)!}{n!}\, h_k,$$

then s has at most $n-k$ exterior facets.

Proof. If s is an n-simplex with facet s' and altitude a over s', then we know from elementary calculus that $V(s) = aV(s')/n$. It is easily verified that the result holds for $n=3$. Assume that it holds for n. If

$$k = n+1 \quad \text{and} \quad V(s) > \frac{n!}{(n+1)!}\, h_n,$$

then s can have no exterior facets since $a \leq 1$ and h_n is an upper bound [5,6] on the volume of an n-simplex in I^n. If s has an exterior facet s', we have

$$\frac{(k-1)!}{(n+1)!}\, h_{k-1} < V(s) \leq \frac{1}{n+1}\, V(s')$$

and s' can have at most $n-k$ exterior facets, by the induction hypothesis. Hence by Lemma 1, s can have at most $(n+1)-k$ exterior facets.

Now we are ready to prove the theorem.

Let S be a triangulation of I^n whose vertices agree with those of I^n and define

$$S_i = \left\{ s \in S\colon \frac{(i-1)!}{n!}\, h_{i-1} < V(s) \leq \frac{i!}{n!}\, h_i \right\} \quad \text{for } 3 \leq i \leq n$$

and

$$S_2 = \{ s \in S\colon V(s) \leq 1/n! \}.$$

Let $a_i = |S_i|$ for $2 \leq i \leq n$. Then

$$\sum_{i=2}^{n} a_i \frac{i!}{n!}\, h_i \geq \sum_{i=2}^{n} V(\bigcup S_i) = V(\bigcup S) = V(I^n) = 1.$$

Since the vertices of S agree with those of I^n, the altitude of a simplex over an exterior facet is 1. Thus if $s \in S$ has two exterior facets s_1 and s_2, we see that $V(s_1) = V(s_2) = nV(s)$. Let S^1 be the set of exterior facets of the simplices in S and define the S_i^1 analogously. The volume of a simplex in S_i^1 is bounded above by $i!\, h_i/(n-1)!$. By Lemmas 2 and 3, $|S_2^1| \leq na_2$ and $|S_i^1| \leq (n-i)a_i$ for $3 \leq i \leq n-1$. Since S^1 is a triangulation of the $2n$ facets of I^n, we have

$$na_2 \frac{2!}{(n-1)!}\, h_2 + \sum_{i=3}^{n-1} (n-i)a_i \frac{i!}{(n-1)!}\, h_i \geq 2n.$$

If we consider the facets of I^n to be disjoint, we can consider two simplices in S^1 as disjoint if they are not contained in the same facet of I^n. Then S^2 can be defined from S^1 as the set of exterior facets of simplices in S^1, so that S^2 is the triangulation of $2n \cdot 2(2n-1) = 2^2 \cdot n!/(n-2)!$ $(n-2)$-cubes. Again by Lemma 3,

$$|S_2^2| \leq (n-1)|S_1^2| = (n-1)na_2 = \frac{n!}{(n-2)!}\, a_2$$

and the maximum volume of a simplex in S_2^2 is $2!\, h_2/n!$. Similarly,

$$|S_i^2| \leq (n-i-1)(n-i)a_i = \frac{(n-i)!}{(n-i-2)!}\, a_i$$

and the maximum volume of a simplex in S_i^2 is $i!/(n-2)!$ for $3 \leq i \leq n-2$. Continuing in this fashion, we have, by a simple induction, the family of inequalities for $n-k$ of the theorem. For $k = n-2$, since $h_2 = \frac{1}{2}$, we see that $a_2 \geq 2^{n-1}$. This proves the theorem.

3. Dimensions 4 and 5

For $n=4$, we have the problem

$$\begin{aligned} \text{minimize} \quad & x_2+x_3+x_4, \\ \text{subject to} \quad & x_2 \geq 8, \\ & 4x_2+2x_3 \geq 48, \\ & x_2+2x_3+3x_4 \geq 24. \end{aligned}$$

It is easily seen that the minimum of 16 is achieved by the four solutions $(8,8,0)$, $(9,6,1)$, $(10,4,2)$, and $(12,0,4)$.

For $n=5$, we have the problem

$$\begin{aligned} \text{minimize} \quad & x_2+x_3+x_4+x_5, \\ \text{subject to} \quad & x_2 \geq 16, \\ & 5x_2+x_3 \geq 120, \\ & 5x_2+4x_3+3x_4 \geq 240, \\ & x_2+2x_3+3x_4+6x_5 \geq 120. \end{aligned}$$

This problem has the solutions $(16,40,0,4)$, $(20,20,20,0)$, $(30,0,30,0)$, and $(48,0,0,12)$ among others for a minimum of 60.

Thus we have a third proof [1,5] that $\varphi(4) \geq 16$ and this is the minimum [3,4]. We also have [5] that $60 \leq \varphi(5) \leq 67$.

4. Remarks

The h_n were chosen simply because they provide an upper bound on $\psi(n)$, the maximum volume of an n-simplex in I^n. It is clear that all the results in this paper still obtain if h_n is replaced by $\psi(n)$ whenever h_n occurs. Since $\psi(5)=5/120$, we could replace the coefficient of x_5 by 5 in the last problem and then solve the system of inequalities.

It is believed that the value of L_n derived in this paper is better than $\frac{1}{2}(h_n+nh_{n-1})$ which was derived in [5]. However it is not known that this is the case.

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